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A quantum harmonic oscillator and strong chaos

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Abstract
It is known that many physical systems which do not exhibit deterministic chaos when treated classically may exhibit such behaviour if treated from the quantum mechanics point of view. In this paper, we will show that an annihilation operator of the unforced quantum harmonic oscillator exhibits distributional chaos as introduced in B Schweizer and J Smítal (1994 *Trans. Am. Math. Soc.* **344** 737–54). Our approach strengthens previous results on chaos in this model and provides a very powerful tool to measure chaos in other (quantum or classical) models.

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In classical mechanics, we work with 'exact' trajectories of points. However, if we follow Heisenberg's uncertainty principle, such trajectories cannot be measured. This is why dynamics is described by the wavefunction obeying some deterministic equation of motion (e.g. Schrödinger equation) in quantum mechanics. In this approach, we work with probability distributions of trajectories rather than with ordinary trajectories themselves. It should be noted that in contrast to stochastic dynamical systems, wavefunctions do not represent distributions of probability (do not have to be positive); however, such distributions may be exactly calculated from wavefunctions (i.e. quantity $|\phi(x,t)|^2$ measures the probability of finding the particle at the position x in time t). It is the phenomena known from the literature that the dynamics of a given system and the dynamics of its probabilistic analogue (e.g. given by the associate Koopman operator acting on a set of densities) may rapidly differ. In fact, it may happen that systems with complicated trajectories will force a very regular motion of the Koopman operator, and a chaotic Koopman operator may correspond to regular dynamics [9]

There exist many different measures of chaos, criteria and definitions. The most common postulates are instability of trajectories (e.g. little change in the accuracy of observation may cause dramatic changes in long-time behaviour) and difficulty in the prediction of trajectories (caused by some kind of mixing present in the system). The list of the most important measures (and definitions) which appear in the classical mechanics may be found in [2] in addition to the very interesting discussion on the possibility of quantum chaos. These notions were usually introduced from the classical mechanics point of view, that is, to analyse properties of deterministic systems. However, as we will see, there is no trouble with their application to quantum mechanics. The only problem is that our intuition gained in the classical mechanics approach may be misleading in this case.

In quantum mechanics, the observables are contained in some Hilbert space of states $\mathscr{H}$ and their dynamics is described by the so-called quantum map, that is, a (usually) unitary operator U_T over some time interval T. This means that for any two observables $\phi, \psi \in \mathscr{H}$, there is no divergence of trajectories:

$$\|\phi_T - \psi_T\| = \|U_T\phi_0 - U_T\psi_0\| = \|U_T(\phi_0 - \psi_0)\| = \|\phi_0 - \psi_0\|, \tag{1}$$

where $\|\cdot\|$ is some norm in $\mathscr{H}$ and ψ_t stands for $\psi_t = \psi(t)$. In most cases the phase space $\mathscr{H}$ is infinitely dimensional, complete but not compact. It implies that the most common measures of chaos such as topological entropy or Lyapunov exponents are all zero. Furthermore, there is no invariant measure for U_T which means that it is not possible to apply other tools known from ergodic theory [1]. Now, it is clear why it is very important to introduce notions of chaos which will be suitable for that case. Two notions which proved their usefulness in physical models are the Devaney (or less restrictive Auslander–Yorke) definition [8, 14] and definition due to Li and Yorke (see [5]). In this paper we will apply another definition of chaos, studied previously only for the compact case, which additionally may be used as a measure of chaos [17] (and is not equivalent to earlier mentioned definitions in general). Studies of nonlinear dynamics are so common in modern science that we usually believe that chaos is a truly nonlinear phenomenon and it is the case for phase spaces embedded in $\mathbb{R}^n$. However, when the dimension is infinite, there is lots of space to produce complicated trajectories [15]. Before we explain how a chaotic phenomenon may arise in a linear harmonic oscillator, we will recall the fundamentals of this model.

The quantum harmonic oscillator is the quantum mechanical analogue of the classical harmonic oscillator. It is one of the most important model systems in quantum mechanics (similar to the importance of a harmonic oscillator in classical mechanics) because of a wide variety of physical situations which can be reduced to it either exactly or approximately. We will give only a brief description of the underlying theory. A more extensive introduction to the topic may be found in books on quantum theory (see [12], [16] or [19]). Evolution of the quantum harmonic oscillator may be modelled by the (time-dependent) Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{m\omega^2}{2}x^2\psi = \mathrm{i}\hbar\frac{\partial\psi}{\partial t} \tag{2}$$

with a wavefunction $\psi(x, t)$, displacement x, mass m, frequency ω and Planck number $\hbar$. For simplicity, we will go over to nondimensional variables (i.e. we set $m = \frac{1}{2}$, $\omega = 2$ and $\hbar = 1$). In this case, (2) takes the form

$$-\frac{\partial^2\psi}{\partial x^2} + x^2\psi = \mathrm{i}\frac{\partial\psi}{\partial t}. \tag{3}$$

One can show using explicit Hilbert space properties that the system of stationary states of the harmonic oscillator $\{\psi_n\}$ forms an orthonormal basis for $\mathscr{H} = L^2(\mathbb{R})$. We recall that a quantum harmonic oscilator may be equivalently described in terms of an annihilation operator $\hat{a} = \frac{1}{\sqrt{2}}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)$ and its adjoint $a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right)$ called a creation operator. By basic properties of Hermite polynomials, we obtain that

$$\hat{a}\psi_n = \frac{1}{\sqrt{2}}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_n = \sqrt{n}\psi_{n-1}. \tag{4}$$

All functions from $\mathscr{H}$ are obviously solutions of (3); however, they cannot describe the resolution of a realistic apparatus (e.g. the detector efficiency). Generally speaking, some solutions from $\mathscr{H}$ are too 'sharp' to appear in nature. This is why we restrict our attention to a subspace Φ of $\mathscr{H}$ defined by the following formula (see [3, p 54 and pp 98–100] for a strict explanation):

$$\Phi = \left\{\phi \in \mathscr{H} : \phi = \sum_{n=0}^{\infty} c_n\psi_n, \sum_{n=0}^{\infty} |c_n|^2 (n+1)^r < \infty \text{ for all } r \geqslant 0\right\}. \tag{5}$$

It is known that Φ is an infinite-dimensional, metrizable, complete and separable topological vector space, and operator $\hat{a}$ is continuous (see [6]). We endow Φ with the standard metric ρ given by the formula

$$\rho(\phi, \psi) = \sum_{m=0}^{\infty} \frac{1}{2^m} \cdot \frac{p_m(\phi - \psi)}{1 + p_m(\phi - \psi)}, \tag{6}$$

where

$$p_m(\phi) = p_m\left(\sum_{n=0}^{\infty} c_n\psi_n\right) = \left(\sum_{n=0}^{\infty} |c_n|^2 (n+1)^m\right)^{1/2}, \qquad m \geqslant 0, \tag{7}$$

is the system of semi-norms defining the topology of Φ. It easily follows from (6) that Φ has diameter bounded by 2.

After we stated the fundamentals of our model, let us recall the notion of distributional chaos. During the study of self-maps of the interval [10], Li and Yorke stated the definition of the phenomenon called chaos for the first time. Their idea was to find an uncountable set D such that for any pair of points $x, y \in D$, the two following conditions hold:

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \tag{8}$$

$$\limsup_{n\to\infty} d(f^n(x), f^n(y) > 0, \tag{9}$$

where d is a metric on a compact space X and $f : X \to X$ is continuous. Condition (8) means that trajectories of these points are arbitrarily close (proximal), but from (9) it means that they are not asymptotic.

Motivated by this study Schweizer and Smítal introduced in [18] the notion of strong chaos, presently known under the name of distributional chaos. Before we state this definition, we must introduce some essential notation.

Let f be a continuous self-map on a metric space (X, d), e.g. $f = \hat{a}$, $X = \Phi$ and $d = \rho$. For any positive integer n, points $x, y \in X$ and $k \in \mathbb{N}$ let

$$\xi(x, y, t, n) = |\{i : d(f^i(x), f^i(y)) < t \quad 0 \leqslant i < n\}|,$$

where $|A|$ denotes the cardinality of set A. We define functions $F_{xy}(t)$ and $F^*_{xy}(t)$ on the real line by

$$F_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n}\xi(x, y, t, n) \tag{10}$$

$$F^*_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n}\xi(x, y, t, n). \tag{11}$$

Both functions F_{xy} and F^*_{xy} are nondecreasing. If diam$X = s < +\infty$, then each of these functions may be naturally viewed as a probability distribution function (it is the case for our set of observables Φ), satisfying $F_{xy}(t) = F^*_{xy}(t) = 0$ for $t < 0$ and $F_{xy}(t) = F^*_{xy}(t) = 1$ for $t > s$. Now, we are ready to state our central definition.

A pair of points $x, y \in X$ is called *distributionally chaotic* if $F_{xy}(s) = 0$ for some $s > 0$ and $F^*_{xy}(t) = 1$ for all $t > 0$. A dynamical system (f, X) is *distributionally chaotic* if there exists an uncountable set D (called a *scrambled set*) such that any pair of distinct points of D is distributionally chaotic. If additionally there exists a parameter $s > 0$, which is good for all pairs of distinct points x, y of D (i.e. $F_{xy}(s) = 0$), then we say that distributional chaos is *uniform*.

Observe that uniform distributional chaos is a very strong chaotic definition. If a system exhibits such kinds of chaos, then there exists constant $\delta > 0$ such that for any two points of the scrambled set, during almost every time step, their iterates are arbitrarily close when looking from one time perspective and almost every iterate of these points is separated by δ when the time perspective is changed.

Distributional chaos implies chaos in the sense of Li and Yorke, as it requires more complicated statistical dependence between orbits than the existence of points which are proximal but not asymptotic. Additionally, distributional chaos (excluding some special cases) is not equivalent to any other kinds of chaos mentioned earlier, e.g. it is possible to construct a dynamical system with zero topological entropy which is distributionally chaotic [11] and systems with positive entropy which do not exhibit distributional chaos [20].

Before we prove that distributional chaos is present in our model, let us say a word about the state of art in the linear model context. The article of Godefroy and Shapiro [7] is one of the first results which shows that Devaney chaos is possible in linear systems (see [15] for a discussion). Gulisashvili and MacCluer applied in [8] the method of Godefroy and Shapiro to prove that the annihilation operator $\hat{a}$ is chaotic in this sense. (Their approach is strongly motivated by a functional analysis.) Strictly speaking, they show that operator $\hat{a}$ fulfils the hypecyclicity criterion and has periodic points dense. The recent results show that operator $\hat{a}$ exhibits even a stronger version of Devaney chaos, as operators fulfilling the hypercyclicity criterion are not only transitive but also exhibit (topological) mixing properties [4]. Duan *et al* proved in [5] that the map $\hat{a}$ is also chaotic in the sense of Li and Yorke. As we will see, these results do not fully reveal a complicated behaviour of the operator $\hat{a}$. Namely, it is possible to construct a scrambled set for $\hat{a}$ and to show that this set is dense and distributional chaos is uniform.

Density of the scrambled set makes us unable to use tools known from a compact case; however we may still follow similar ideas. During our construction, we will combine methods from [5] and [13]. The construction we perform is somehow technical in nature; however, it may be effectively applied to similar models which arise naturally in quantum physics.

At the beginning, we have to construct some important sequence of times inductively. Let $S^1_1 = 2$, let us set any $l \geqslant 1$ and let us assume that S^i_j is defined for all $0 < i \leqslant l$ and $j = 1, \ldots, i$. Let

$$S^{l+1}_1 = 2^l \sum_{p=1}^{l} \sum_{q=1}^{p} S^p_q. \tag{12}$$

If S^{l+1}_k is defined for some $1 \leqslant k \leqslant l$, we define S^{l+1}_{k+1} by the formula

$$S^{l+1}_{k+1} = 2^l \left(\sum_{p=1}^{l} \sum_{q=1}^{p} S^p_q + \sum_{r=1}^{k} S^{l+1}_r \right). \tag{13}$$

Thus, S_k^l is defined for any $l \geqslant 1$ and $1 \leqslant k \leqslant l$. We also use the symbol M_k^{l+1} to denote the sum of sequence S_q^p, that is,

$$M_k^{l+1} = \sum_{p=1}^{l} \sum_{q=1}^{p} S_q^p + \sum_{r=1}^{k} S_r^{l+1}. \tag{14}$$

At this point, we are ready to start the construction of the scrambled set $D \subset \Phi$. Let J denote an open interval $J = (\alpha, \beta)$, where α, β are any constants fulfilling $0 < \alpha < \beta < 1$. Let us fix any $\theta \in J$. We define the family $\phi^\theta = \sum_{n=0}^{\infty} c_n^\theta \psi_n \in \Phi$, where $c_0^\theta = 0$ and

$$c_n^\theta = \begin{cases} \dfrac{1}{\sqrt{n!}} & \text{if} \quad n \in \left[M_k^l, M_{k+1}^l\right) \quad \text{and} \quad \theta \geqslant \dfrac{k+1}{l}, \qquad 1 \leqslant k < l, l \geqslant 1 \\ \dfrac{1}{\sqrt{n!}} & \text{if} \quad n \in \left[M_l^l, M_1^{l+1}\right) \quad \text{and} \quad \theta \geqslant \dfrac{1}{l+1} \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Let $D = \{\phi^\theta : \theta \in J\}$, let us set any $\varepsilon > 0$ and let $\theta_1, \theta_2 \in J$. There exist positive integers K, ω, N such that

$$\sum_{i=K+1}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{2}, \qquad \sum_{m=\omega}^{\infty} \frac{(m+1)^K}{2^m} < \left(\frac{\varepsilon}{6}\right)^2, \qquad \frac{N}{N+1} > \beta, \qquad 2^N > \omega. \tag{16}$$

For any $n > N$, let us define $i_n = M_n^{n+1}$, $t_n = M_{n+1}^{n+1}$. By (15) we obtain that $c_j^{\theta_1} = 0 = c_j^{\theta_2}$ for $i_n \leqslant j < t_n$ because $\theta_1 < 1$ and $\theta_2 < 1$ and (16) implies that $i_n < t_n - \omega$. Let us take any integer s such that $i_n \leqslant s < t_n - \omega$. Then,

$$\rho(\hat{a}^s(\phi^{\theta_1}), \hat{a}^s(\phi^{\theta_2})) \leqslant \frac{\varepsilon}{2} + \sum_{m=0}^{K} \frac{1}{2^m} \cdot \frac{p_m(\hat{a}^s(\phi^{\theta_1} - \phi^{\theta_2}))}{1 + p_m(\hat{a}^s(\phi^{\theta_1} - \phi^{\theta_2}))} \tag{17}$$

and additionally, for $r < K$, the following inequalities hold:

$$\begin{aligned} p_r(\hat{a}^s(\phi^{\theta_1} - \phi^{\theta_2})) &\leqslant \left(\sum_{m=\omega}^{\infty} \frac{(m+1)^K}{m!}\right)^{\frac{1}{2}} \\ &\leqslant \left(\sum_{m=\omega}^{\infty} \frac{(m+1)^K}{2^m}\right)^{\frac{1}{2}} < \frac{\varepsilon}{6}. \end{aligned} \tag{18}$$

Combining (17) and (18), we obtain

$$\rho(\hat{a}^s(\phi^{\theta_1}), \hat{a}^s(\phi^{\theta_2})) \leqslant \frac{\varepsilon}{2} + \sum_{m=0}^{K} \frac{\varepsilon}{6 \cdot 2^m} = \frac{\varepsilon}{3} + \frac{\varepsilon}{2} < \varepsilon. \tag{19}$$

Then by the choice of s and (19), we have

$$\begin{aligned} \frac{1}{t_n} \xi(\phi^{\theta_1}, \phi^{\theta_2}, \varepsilon, t_n) &\geqslant \frac{S_{n+1}^{n+1} - \omega}{M_n^{n+1} + S_{n+1}^{n+1}} \\ &\geqslant \frac{2^{n+1} M_n^{n+1} - \omega}{M_n^{n+1} + 2^{n+1} M_n^{n+1}} \longrightarrow 1. \end{aligned} \tag{20}$$

Following (20) we see that $F^*_{\phi^{\theta_1}\phi^{\theta_2}}(\varepsilon) = 1$ for any distinct $\phi^{\theta_1}, \phi^{\theta_2} \in D$ and $\varepsilon > 0$.

To finish the proof that D is the scrambled set (and distributional chaos is uniform) for $\hat{a}$, it is enough to show that $F^*_{\phi^{\theta_1}\phi^{\theta_2}}\left(\frac{1}{16}\right) = 0$. Let us set an integer N which is large enough to have $|\theta_1 - \theta_2| > \frac{4}{N}$, $|1 - \theta_2| > \frac{4}{N}$ and assume that $\theta_2 > \theta_1$. For any $n > N$ there exists

$1 < k < n$ such that $\theta_1 < \frac{k}{n+1} < \theta_2$. Let $\tilde{i}_n = M_k^{n+1}$, $\tilde{t}_n = M_{k+1}^{n+1}$ and let us set any j such that $\tilde{i}_n \leqslant j < \tilde{t}_n$. In this setting, $c_j^{\theta_1} = 0$ and $c_j^{\theta_2} = \frac{1}{\sqrt{j!}}$, which yields that

$$p_1(\hat{a}^j(\phi^{\theta_1} - \phi^{\theta_2})) \geqslant \left(\left|c_j^{\theta_1} - c_j^{\theta_2}\right|^2 \cdot j!\right)^{\frac{1}{2}} = 1 \tag{21}$$

and from the other side

$$p_1(\hat{a}^j(\phi^{\theta_1} - \phi^{\theta_2})) \leqslant \left(\sum_{m=0}^{\infty} \frac{m+1}{m!}\right)^{\frac{1}{2}} \leqslant \left(1 + \sum_{m=1}^{\infty} \frac{1}{2^{m-1}}\right)^{\frac{1}{2}} \leqslant 3. \tag{22}$$

By virtue of (21) and (22), the following inequality holds:

$$\rho(\hat{a}^j(\phi^{\theta_1}), \hat{a}^j(\phi^{\theta_2})) \geqslant \frac{1}{2} \cdot \frac{p_1(\hat{a}^j(\phi^{\theta_1} - \phi^{\theta_2}))}{1 + p_1(\hat{a}^j(\phi^{\theta_1} - \phi^{\theta_2}))} \geqslant \frac{1}{8} > \frac{1}{16}, \tag{23}$$

and as a result

$$\frac{1}{\tilde{t}_n} \xi\left(\phi^{\theta_1}, \phi^{\theta_2}, \frac{1}{16}, \tilde{t}_n\right) \leqslant \frac{\tilde{i}_n}{\tilde{t}_n} = \frac{M_k^{n+1}}{M_k^{n+1} + S_{k+1}^{n+1}} = \frac{1}{2^{n+1} + 1} \longrightarrow 0. \tag{24}$$

This proves that D is the scrambled set for $\hat{a}$, and distributional chaos is uniform. Furthermore, we may choose D to be dense because Φ is separable, and we may include an arbitrarily large (but finite) number of scaled stationary states in our wave packet without interrupting long-time behaviour of the annihilation operator $\hat{a}$. Additionally, our approach may be used to measure chaos. In [17] the following measure was introduced:

$$\mu_p(f) = \sup_{x,y \in X} \frac{1}{\mathrm{diam} X} \int_0^{\infty} F_{xy}^*(t) - F_{xy}(t)\, \mathrm{d}t. \tag{25}$$

It is a hard task to calculate this supremum; however, positiveness of $\mu_p(f)$ may be used as a preliminary test for chaos. This approach is more nice than that of topological entropy where (usually easier to calculate) upper bound does not help to answer whether the system is chaotic or not. In our case,

$$\mu_p(\hat{a}) \geqslant \sup_{\phi,\psi \in D} \frac{1}{2} \int_0^{\frac{1}{16}} F_{\phi\psi}^*(t) - F_{\phi\psi}(t)\, \mathrm{d}t = \frac{1}{16}, \tag{26}$$

which once again ensures us about the chaocity of the model. It would be interesting to know the exact value of $\mu_p(\hat{a})$ or to compute this quantity in other physical models (e.g. given by the Schrödinger equation).

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